stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZW 96/77 APRIL
(DEPARTMENT OF PURE MATHEMATICS)

J. VAN DE LUNE

A NOTE ON NONVANISHING FOURIER TRANSFORMS

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 42A68, 40E05.

A note on nonvanishing Fourier transforms

by

J. van de Lune

ABSTRACT

It is shown that if $\phi \in L^1(\mathbb{R}^+)$, ϕ is monotonically non-increasing on $\mathbb{R}^+$ whereas there exists some interval on which ϕ is strictly decreasing then the Fourier transform $\hat{\phi}$ of ϕ has no real zeros.

Moreover it is shown that the condition "there exists some interval on which ϕ is strictly decreasing" is not a necessary condition in order to have $\hat{\phi}(t) \neq 0$, $\forall t \in \mathbb{R}$.

KEY WORDS & PHRASES: *Fourier transforms, Tauberian theorems.*

1. Recently L.A. RUBEL [1] proposed the following problem: Prove that if $\phi \in L^1 := L^1(\mathbb{R})$, $\phi \geq 0$, $\phi(x) = 0$ for all x outside an interval $[a,b]$ and ϕ is strictly decreasing on $[a,b]$ then the span of all translates of ϕ is dense in L^1.

By Wiener's general Tauberian theorem for L^1 (c.f. [2; p.9]) the span of all translates of some $\phi \in L^1$ is dense in L^1 if and only if the Fourier transform $\hat{\phi}$ of ϕ has no real zeros.

From this it is clear that RUBEL's result is an immediate consequence of the following more general

THEOREM 1. *If $\phi \in L^1$, $\phi(x) = 0$ for all $x < a$ and ϕ is monotonically non-increasing on $[a,\infty)$ whereas there exists some interval $[\alpha,\beta]$ with $\alpha < \beta$, on which ϕ is strictly decreasing, then the Fourier transform $\hat{\phi}$ of ϕ has no real zeros.*

PROOF. Without loss of generality we may assume that $a = 0$. From the conditions in the theorem it is clear that

(1) $\qquad \phi(x) \geq 0 \quad$ for all $\quad x \geq 0$,

(2) $\qquad \phi(x) < \infty \quad$ for all $\quad x > 0$,

(note that $\phi(+0)$ is possibly not finite)

(3) $$\int_0^\infty \phi(x)\,dx > 0,$$

(4) $$\lim_{x\to\infty} \phi(x) = 0,$$

(5) $\qquad 0 \leq \alpha < \beta$.

We proceed by contradiction. If the theorem is false then there exists a $t_0 \in \mathbb{R}$ such that

(6) $$\hat{\phi}(t_0) = \int_{\mathbb{R}} e^{it_0x}\phi(x)\,dx = \int_0^\infty e^{it_0x}\phi(x)\,dx = 0.$$

From (3) it is clear that $t_0 \neq 0$ and since ϕ is real we have

(7) $$\overline{\hat{\phi}(t_0)} = \hat{\phi}(-t_0)$$

so that we may assume that $t_0 > 0$.
Now choose any $\theta > 0$ and observe that

(8) $$0 = \hat{\phi}(t_0) = \{ \int_0^\theta + \int_\theta^\infty \} e^{it_0x} \phi(x)\, dx =$$

$$= \int_0^\theta e^{it_0x} \phi(x)\, dx + \frac{1}{it_0} \int_\theta^\infty \phi(x)\, d\, e^{it_0x} =$$

$$= \int_0^\theta e^{it_0x} \phi(x)\, dx + \frac{1}{it_0} \{ \phi(x)\, e^{it_0x} \Big|_{x=\theta}^{x=\infty} - \int_\theta^\infty e^{it_0x}\, d\phi(x) \} =$$

(since $\lim_{x\to\infty} \phi(x) = 0$)

$$= \int_0^\theta e^{it_0x} \phi(x)\, dx - \frac{1}{it_0} \phi(\theta)\, e^{i\theta t_0} - \frac{1}{it_0} \int_\theta^\infty e^{it_0x}\, d\phi(x).$$

It follows that

(9) $$-it_0\, e^{-i\theta t_0} \int_0^\theta e^{it_0x} \phi(x)\, dx + \phi(\theta) = -\int_\theta^\infty e^{it_0(x-\theta)}\, d\phi(x).$$

Defining

(10) $$h(\theta) = -it_0\, e^{-i\theta t_0} \int_0^\theta e^{it_0x} \phi(x)\, dx, \quad (\theta > 0)$$

and

(11) $$h^*(\theta) = \mathrm{Re}\, h(\theta), \quad (\theta > 0)$$

we have

(12) $$h^*(\theta) + \phi(\theta) = \int_\theta^\infty \cos t_0(x-\theta)\, d\Psi(x)$$

where $\Psi = -\theta$.

Since $t_0 > 0$ we thus obtain by the substitution $t_0(x-\theta) = u$

(13) $$h^*(\theta) + \phi(\theta) = \int_0^\infty \cos u\, d\Psi(\theta + \frac{u}{t_0}) \ .$$

Now observe that the (non-degenerate) interval

(14) $$[\alpha t_0, \beta t_0] \subset [0,\infty)$$

contains a subinterval $[\gamma,\delta]$, say, such that for some $\varepsilon > 0$

(15) $$\cos u \leq 1 - \varepsilon \quad \text{for all} \quad u \in [\gamma,\delta] \ .$$

Since $\Psi = -\phi$ is non-decreasing on $[\theta,\infty)$ it follows that

(16) $$h^*(\theta) + \phi(\theta) = \{ \int_0^\gamma + \int_\gamma^\delta + \int_\delta^\infty \} \cos u\ d\ \Psi(\theta + \frac{u}{t_o}) \leq$$

$$\leq \int_0^\gamma d\ \Psi(\theta + \frac{u}{t_o}) + (1-\varepsilon) \int_\gamma^\delta d\ \Psi(\theta + \frac{u}{t_0}) + \int_\delta^\infty d\ \Psi(\theta + \frac{u}{t_0}) =$$

$$= \int_0^\infty d\ \Psi(\theta + \frac{u}{t_0}) - \varepsilon \int_\gamma^\delta d\ \Psi(\theta + \frac{u}{t_0}) =$$

$$= -\Psi(\theta) - \varepsilon \int_\gamma^\delta d\ \Psi(\theta + \frac{u}{t_0})$$

and consequently

(17) $$h^*(\theta) \leq \varepsilon\{\Psi(\theta + \frac{\gamma}{t_0}) - \Psi(\theta + \frac{\delta}{t_0})\} \ .$$

Since

$$(18) \qquad |h(\theta)| \le t_0 \int_0^\theta \phi(x)\,dx$$

we have

$$(19) \qquad \lim_{\theta \downarrow 0} h(\theta) = 0$$

so that certainly

$$(20) \qquad \lim_{\theta \downarrow 0} h^*(\theta) = 0.$$

Hence, taking limits for $\theta \downarrow 0$ we obtain from (17) that

$$(21) \qquad 0 \le \varepsilon\{\Psi(\frac{\gamma}{t_0} + 0) - \Psi(\frac{\delta}{t_0} + 0)\}$$

and, since $\varepsilon > 0$, it follows that

$$(22) \qquad \phi(\frac{\gamma}{t_0} + 0) \le \phi(\frac{\delta}{t_0} + 0) \quad .$$

However, observing that

$$(23) \qquad \alpha \le \frac{\gamma}{t_0} < \frac{\delta}{t_0} \le \beta$$

and that ϕ is strictly decreasing on $[\alpha,\beta]$ we must have

$$(24) \qquad \phi(\frac{\gamma}{t_0} + 0) > \phi(\frac{\delta}{t_0} + 0) \quad .$$

This contradicts (22) and completes the proof.

2. In theorem 1, the condition "there exists some (non-degenerate) interval $[\alpha,\beta]$ on which ϕ is strictly decreasing" is not a necessary condition as may be seen from the following

THEOREM 2. *Let* $\{a_n\}_{n=-\infty}^{\infty}$ *be a strictly decreasing sequence such that* $\lim_{n\to\infty} a_n = 0$. *In addition let* $\{w_n\}_{n=-\infty}^{\infty}$ *be a monotonically non-decreasing sequence such that* $\lim_{n\to\infty} w_n = \infty$. *Define* $\phi : \mathbb{R} \to \mathbb{R}$ *by*

$$(2.1) \qquad \phi(x) = \begin{cases} 0 & \text{if } x \le 0 \\ w_n & \text{if } a_{n+1} < x \le a_n \end{cases}$$

and let $\phi \in L^1$, i.e.

$$(2.2) \qquad \sum_{n=-\infty}^{\infty} w_n(a_n - a_{n+1}) < \infty \quad .$$

Then the Fourier transform $\hat{\phi}$ *of* ϕ *has no real zeros.*

PROOF. Suppose the theorem is false. Then there exists a $t_0 \in \mathbb{R}$ such that $t_0 > 0$ and

$$(2.3) \qquad 0 = \hat{\phi}(t_0) = \int_0^{\infty} e^{it_0x}\phi(x)\,dx = \sum_{n=-\infty}^{\infty} w_n \int_{a_{n+1}}^{a_n} e^{it_0x}\,dx =$$

$$= \frac{1}{it_0} \sum_{n=-\infty}^{\infty} w_n(e^{it_0a_n} - e^{it_0a_{n+1}})$$

so that

$$(2.4) \qquad \sum_{n=-\infty}^{\infty} w_n\{\cos(t_0a_n) - \cos(t_0a_{n+1})\} = 0$$

or, equivalently

(2.5) $$\lim_{M,N\to\infty} \sum_{n=-M}^{N} w_n\{\cos(t_0 a_n) - \cos(t_0 a_{n+1})\} = 0.$$

Setting $c_k = \cos(t_0 a_k) - 1$ we have

(2.6) $$\sum_{n=-M}^{N} w_n\{\cos(t_0 a_n) - \cos(t_0 a_{n+1})\} =$$

$$= \sum_{n=-M}^{N} w_n(c_n - c_{n+1})$$

which, by summation by parts,

$$= w_{-M}c_{-M} + \sum_{n=-M+1}^{N} c_n(w_n - w_{n-1}) - w_N c_{N+1} \quad .$$

Since $\phi \in L^1$ it is clear that $\lim_{N\to\infty} w_{-N} = 0$ and thus, since $|c_{-n}| \le 2$,

(2.7) $$\lim_{N\to\infty} w_{-N}c_{-N} = 0 \quad .$$

We shall now show that

(2.8) $$\lim_{M\to\infty} w_M c_{M+1} = 0 \quad .$$

We have

(2.9) $$w_M c_{M+1} = w_M\{\cos(t_0 a_{M+1}) - 1\} =$$

$$= \frac{\cos(t_0 a_{M+1}) - 1}{(t_0 a_{M+1})^2} \, t_0^2 \, (w_M a_{M+1}^2)$$

and since

$$\text{(2.10)} \qquad \lim_{M\to\infty} t_0 a_{M+1} = 0$$

and

$$\text{(2.11)} \qquad \lim_{x\to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}$$

it suffices to show that

$$\text{(2.12)} \qquad \lim_{M\to\infty} w_M a_{M+1}^2 = 0 \quad .$$

Since

$$\text{(2.13)} \qquad \lim_{M\to\infty} a_M = 0 \quad \text{and} \quad 0 < a_{N+1} < a_N$$

it clearly suffices to show that $w_M a_M$ is bounded for $M \to \infty$. Using the monotonicity of w_n we obtain

$$\text{(2.14)} \qquad w_M a_M = w_M \sum_{n=M}^{\infty} (a_n - a_{n+1}) =$$

$$= \sum_{n=M}^{\infty} w_M (a_n - a_{n+1}) \leq \sum_{n=M}^{\infty} w_n (a_n - a_{n+1}) \leq \int_0^{\infty} \phi(x)\, dx$$

and (2.8) follows.

Hence

$$\text{(2.15)} \qquad 0 = \sum_{n=-\infty}^{\infty} c_n (w_n - w_{n-1}) = \sum_{n=-\infty}^{\infty} \{\cos(t_0 a_n) - 1\} (w_n - w_{n-1}) \quad .$$

Plainly each term in the above series is ≤ 0 and hence $= 0$. However, $\cos(t_0 a_n) < 1$ for all sufficiently large n and thus, since $\lim_{n\to\infty} w_n = 0$, there exists at least one term in the above series which is non zero. This contradiction completes our proof.

If we replace the condition "$\lim_{n\to\infty} w_n = \infty$" by the condition "$w_n$ is strictly increasing" then we obtain from (2.15) that $\cos(t_0 a_n) - 1 = 0$ for all $n \in \mathbb{N}$ so that

$$(2.16) \qquad (0 <)\ t_0 a_n = k_n \cdot 2\pi \qquad \text{for some } k_n \in \mathbb{Z}$$

contradicting our assumption that $\lim_{n\to\infty} a_n = 0$.

Finally, we may also replace the condition "w_n is strictly increasing" by the condition "w_n is non-decreasing and $w_n < \lim_{m\to\infty} w_m$ for all $n \in \mathbb{Z}$".

We thus obtain the following

THEOREM 3. *If the stepfunction* $\phi : (0,\infty) \to \mathbb{R}$, *(in the sense of theorem 2) belongs to* L^1 *and is monotonically non-increasing and assumes infinitely many different values in every neighborhood of* 0, *then* $\hat{\phi}(t) \neq 0$ *for all* $t \in \mathbb{R}$.

The condition "assumes infinitely many different values in every neighborhood of 0" cannot be deleted in general as may be seen from a simple example such as: $\phi(x) = 1$ if $x \in (0,1)$ and $\phi(x) = 0$ if $x \notin (0,1)$.

From the above considerations it also follows that the Fourier transform of a non-increasing stepfunction (in the sense of theorem 2) can have zeros only when all its jumps take place within a set consisting of the multiples of some positive number.

REFERENCES

[1] RUBEL, L.A., *Advanced Problem 6131*, Amer. Math. Monthly (48)1(1977), p. 62.

[2] WIENER, N.,*Tauberian theorems*, Ann. Math. 33(1932), pp. 1-100.